An APOS Analysis of Student Learning of Congruence in Taxicab Geometry

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Abstract

Non-Euclidean geometries are commonly used in the education of college geometry students to enhance their understanding of Euclidean geometry. Of these geometries, Taxicab geometry can be introduced to help students generalize their current understandings of specific concepts in Euclidean geometry. Of these concepts, Taxicab geometry perturbs students' definition of congruence. This research project is part of a larger research project that aims to encourage students to define congruence with respect to isometries. Using the APOS framework, data collected from classroom participation was analyzed to investigate how students would develop this definition in Taxicab geometry.

Keywords: Taxicab geometry, APOS, Congruence

Introduction

Commonly, when students explore congruence in geometry, they investigate three triangle congruence criteria, Side-Angle-Side (SAS), Side-Side-Side (SSS), and Angle-Angle-Side (AAS). These three criteria for triangle congruence have been proven true mathematically in Euclidean geometry; however, the congruence criteria are not always true if one considers objects in other geometries. For example, two triangles may have congruent corresponding sides in a given geometry so they satisfy the SSS criteria, however, the triangles may not have congruent angles. Taxicab geometry is often introduced and developed in college geometry courses so that students unpack certain geometric concepts that they may have grown to misunderstand due to considering them only in Euclidean geometry. Of those concepts that have to be unpacked are the three triangle congruence criteria and what it even means for two geometric objects to be congruent in the first place. A more robust definition of congruence than what is found in *Elements* (Euclid and Heath, 2002) is that two objects are congruent if one is able to superimpose one object onto another through executing a series of rigid motions

(motions in geometry that do not change the constituent parts of a geometric figure). Unpacking this definition is one aim of introducing Taxicab geometry to college geometry students. Through the Action-Process-Object-Schema (APOS) framework (Dubinsky, 2002), a greater understanding could be reached about how college geometry students define congruence with respect to isometries. This study seeks to answer how students' thinking about congruence in Taxicab geometry develop.

Literature Review

Action-Process-Object-Schema Framework

APOS theory is based on Jean Piaget's theory of reflective abstraction, which is "the construction of mental objects and of mental actions on these objects" (Dubinsky, 2002, p. 102). APOS theory is made up of four different levels of cognitive development, Action, Process, Object, and Schema. A student is thinking at an Action level when performing actions to manipulate objects (Dubinsky, 2002). Such objects can be numbers and examples of actions that can be used to manipulate these numbers are multiplication, addition, subtraction amongst others. A student can then construct a process through interiorizing these actions (Dubinsky). A function can be considered as a process when there is an understanding that an input will possibly have some changes resulting in an output without the need to perform operations on this input. The comprehension of a function as a process in this manner demonstrates that a student is understanding functions in the Process level. To clarify, a student can also understand functions in the Action level if their comprehension of a function is a series of operations that must be performed. Additionally, a student can also construct processes through the coordination of other processes or even through the reversal of other processes such as the derivative and the integral (Dubinsky). It is the encapsulation of processes that creates objects (Dubinsky). Continuing with the example of functions, functions can also be understood as objects and thereby, a student is understanding them at the Object level. Functions are treated as objects when actions are imposed on them, such as in function composition. The Schema level refers to the coherent collection of objects and processes of a concept (Dubinsky). This schema can also have schemas of other concepts collected when those two or more concepts are related to them, such as the schema of operations being related to the schema of a function or the schema of the radius of a circle being related to the schema of a circle (Kemp and Vidakovic, 2017).

These concepts do not necessarily have to be understood in a single level at a time. A function, for example, can be understood in all three levels at once. Through composition, a student may first understand functions to be objects where one function is the input to another one but then one of the functions has to be unpacked into all the actions that must be performed. The result and continuation of practice in function composition, can allow a person to internalize compositions as a process.

Taxicab Geometry

Taxicab geometry is best discussed in relation to Euclidean geometry (Kraus, 1986). Most students are exposed to Euclidean geometry throughout formal education at the K-12 level; however, the name, Euclidean geometry, is not distinguished due to the lack of exposure to other geometries. So to distinguish Euclidean geometry, it is important to know that Euclidean geometry has a set of axioms and a metric (the way distance is measured); one can create a non-Euclidean geometry by changing either the axioms or the metric. For example, changing Playfair’s axiom (an axiom of Euclidean geometry) can generate hyperbolic or elliptical geometry; these new geometries are known as Non-Euclidean geometries. An easily-accessible Non-Euclidean geometry is Taxicab geometry, which is constructed by changing the metric of Euclidean geometry.

The best way to illustrate the difference in metrics of Euclidean geometry and Taxicab geometry is through their different approaches to calculating distance. If one imposes the standard Cartesian coordinate system Euclidean distance between two points, A and B, is measured by the equation

$$d_E(A,B) := \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \quad (1)$$

where point *A* is the ordered pair (x_1, y_1) and *B* is the ordered pair(x_2, y_2). Taxicab distance between two points, A and B, is measured by the equation

$$d_T(A,B) := |x_2 - x_1| + |y_2 - y_1| \quad (2)$$

where point *A* is the ordered pair (x_1, y_1) and *B* is the ordered pair(x_2, y_2). These differences can both be seen in Figure 1.

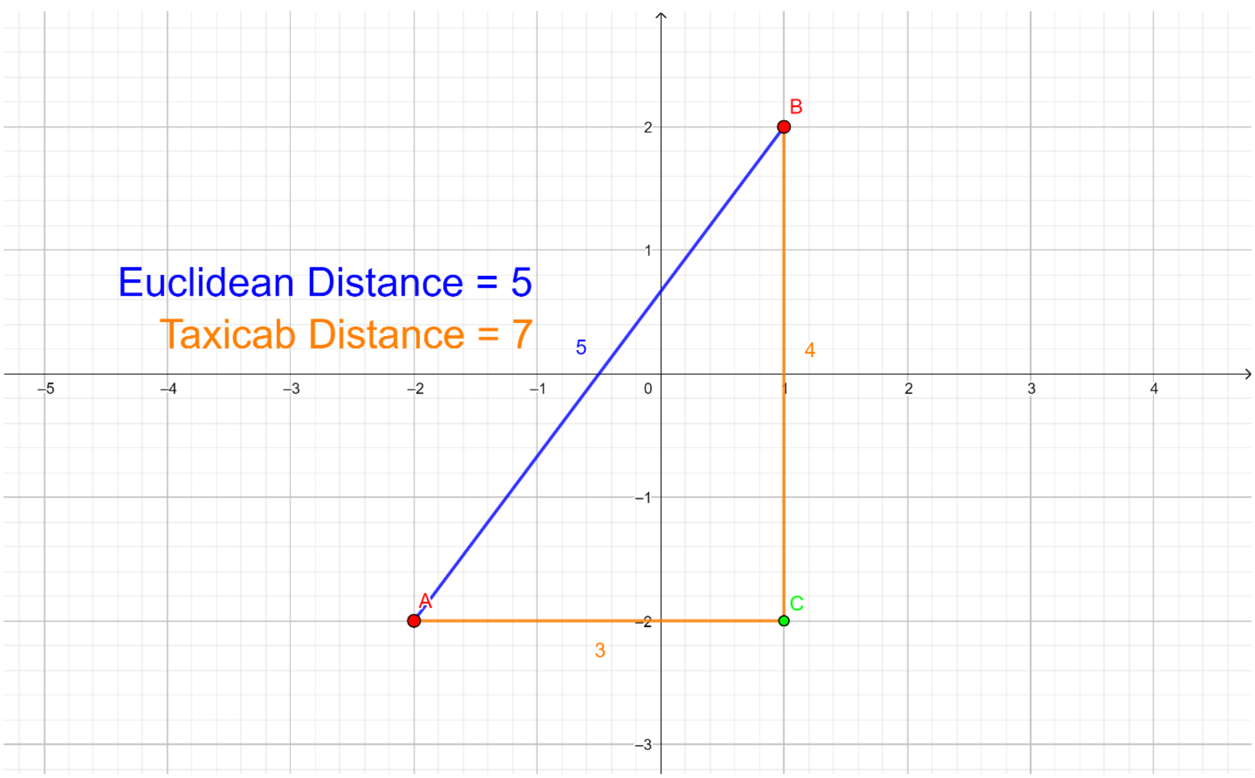

Figure 1: Euclidean Distance and Taxicab Distance Comparison

While it may seem that the difference between Euclidean and Taxicab geometry is small, there are fundamental differences in the way shapes and concepts are understood. For example, a circle in Taxicab geometry will actually appear to be a square rather than the shape people have grown to expect of a circle (see Figure 2).

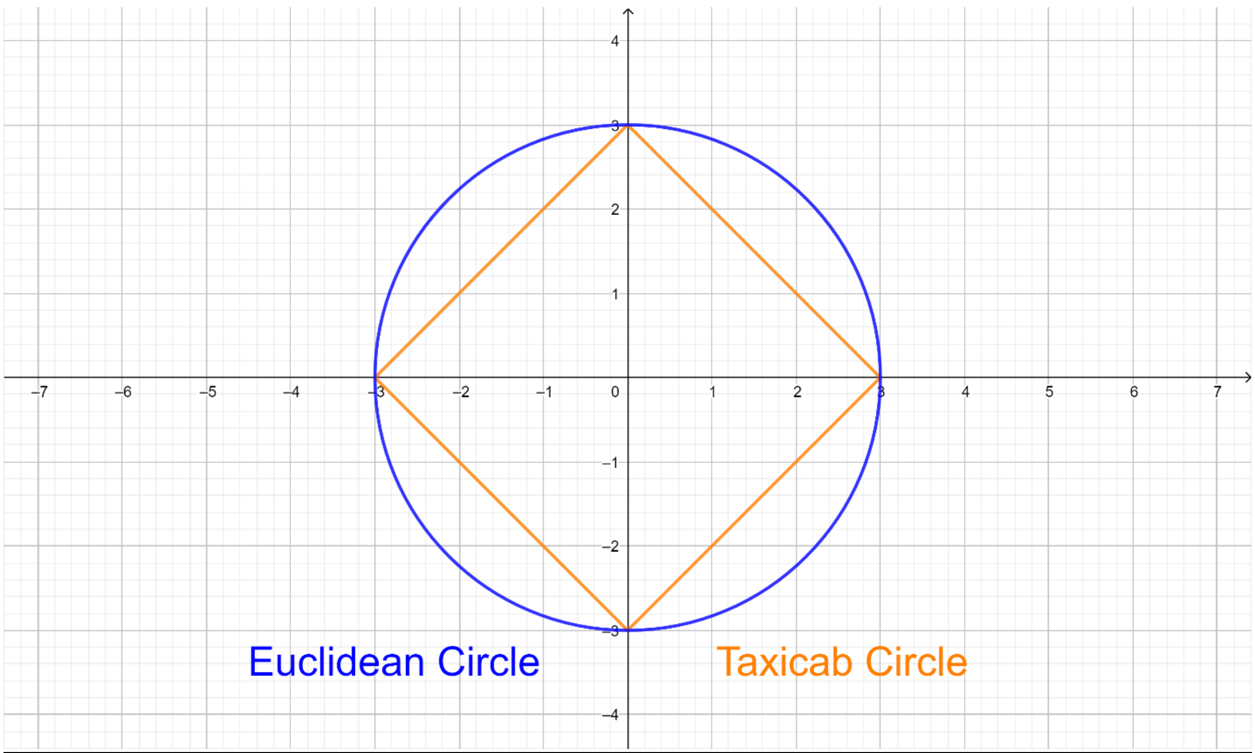

Figure 2: Euclidean Circle vs. Taxicab Circle

Using the APOS Framework to Study Taxicab Geometry

Recently, Kemp and Vidakovic (2017) studied how a student uses the definition of a circle to derive the Taxicab circle equation. In this study, the researchers constructed an expected trajectory consisting of concepts related to a Taxicab circle, such as distance, and the representation of a Euclidean circle along with the APOS levels. Additionally, Kemp (2018) completed her dissertation research on how students transfer mathematical definitions from Euclidean geometry to Taxicab geometry. In this study, Kemp followed how 15 students developed their schema of certain concepts in either geometry and how it is that this development helped in their understanding of the concept in multiple geometries as well.

Methodology

The data was collected from 41 students in a college geometry course at a large southwestern university. Most of the students required to take this course were prospective middle and high school mathematics teachers while other students enrolled were mathematics majors who took the course as an elective. An introduction to proof writing course is not a pre-requisite, although many students had completed this course before taking college geometry.

The instructor introduced the Side-Angle-Side (SAS) triangle congruence criterion as a missing postulate in Euclidean geometry in this course, which is a standard pedagogical move when discussing triangle congruence criteria. Students explored the uses of the SAS congruence criterion by using it to prove other triangle congruence criteria, as well as the Isosceles Triangle theorem and properties of quadrilaterals. The instructor also introduced transformational geometry with an emphasis on the definition of isometries. (An *isometry* is a transformation of the plane that preserves distance.) Additionally, students studied fixed point and orientation properties, results of composing isometries, and analytic formulas for particular plane isometries in the Cartesian plane.

Students also took part in a week-long investigation of Taxicab geometry. Following the introduction of the Taxicab metric, the students studied relationships between Euclidean and Taxicab geometry by completing the following tasks (Table 1). Student work in small groups on these tasks was recorded to gather information of how participants understood Taxicab geometry.

Table 1: Taxicab geometry Tasks (P.V. Prasad, personal communication, May 2018)

1) Come up with examples of each, or explain why such an example is not possible
    a) 2 triangles that are congruent in both Euclidean geometry and Taxicab geometry
    b) 2 triangles that are congruent in Euclidean geometry but **not** Taxicab geometry
    c) 2 triangles that are congruent in Taxicab geometry but **not** Euclidean geometry
2) Identify all the Taxicab isometries.
3) How can we define congruence in Taxicab geometry?

Students worked on these problems while in 10 assigned groups of about four students per group. Nine of the groups were audio recorded using Livescribe dotted paper and smart pens to digitize their writing so that it would be in sync with their audio recording. The Livescribe dot paper is standard printed paper with a unique pattern of dots so that the smart pen can capture the exact location of everything that it writes or draws, allowing the smart pen to digitize what has been written at any given time while the pen recorded and synchronized audio as well ("Dot Paper," n.d.). It was through this technology that the researcher kept track of what was written at what time and what was being discussed during that time period as well.

After taking notes on the recording for each group, groups one, eight, and nine were selected for transcription due to the richness of mathematical thinking that they provided about solving the assigned tasks. The researcher then listened to the group recordings and writings again, reading through the transcriptions as well searching for the development of a definition of congruence with respect to isometries and determining what type of APOS understanding the students demonstrated in their reasoning.

Results

Group One

Students in group one answered Task 1a by noticing that a pair of right triangles (when oriented in the same manner) would be congruent in both geometries. The students displayed that their concept of congruence of triangles was in the Process level since they had no need to verify this fact through measuring sides or angles but through having internalized the manner in which one measures distance in either geometry. Continuing from this task, the students considered what types of triangles in either geometry would not be congruent if paired. In other words, the students would measure and then compare two right, isosceles, scalene, equilateral, etc. triangles in either geometry expecting to find a difference in congruence. This example supports the observation that the students believed congruence was dependent on the type of triangle they were working with as opposed to how isometries were being used to determine congruence; however, after considering different types of triangles, the students actually attempted to find out what would happen to a triangle if it were to “move” to a different position. The students conjectured that the position or orientation of a triangle on a plane would affect the distance of some of the sides of the triangle. Since the group needed to verify the truth of this conjecture, by making an example, the students displayed that their concept of congruence had to be reverted to the Action level in order to unpack their previous Euclidean interpretation of congruent triangles. For the rest of the class period, as the group moved on to Task 1c (Table 1), they once again started trying to use specific types of triangles to help them solve Task 1c without, once again, considering the orientation. It appears that the students may have believed that congruence depends on the type of triangles being used, and since they had the need to confirm through examples, it may be determined that the students were still at the Action level of their congruence concept, but that the concepts were at the moment changing and the group was still analyzing them.

While continuing their work on Task 1c, one of the students became aware that in Euclidean geometry, two objects may be oriented differently and still be congruent but the objects' congruence could change in Taxicab geometry. This sole student had internalized how orientation would affect distance and congruence in either geometry, displaying a Process level for the concept of congruence but relying on using visual examples to help group members come to the same conclusion. This use of examples convinced the entire group that orientation was important when thinking about congruence; therefore, the rest of the group was at the Action level for orientation with respect to congruence. To conclude this task, another student figured that an example for Task 1c cannot exist because Taxicab isometries are subsets of Euclidean isometries. This student claimed that isometries are a part of a set and, therefore, the student had an Object level of understanding the concept of isometries.

The students left Task 1c with that claim in mind and decided that they should move on to Task 2. In this second task, the students were required to consider specific transformations, so they reverted to the Object level to consider transformations at either the Process or Action level. It was readily clear to the group that translation (sliding in a graph while preserving orientation) in Taxicab geometry did not have to be tested to know if it is an isometry in Taxicab geometry since they had already internalized that translations do not affect orientation and, therefore, do not affect distance either. Due to their Process level of translations, the students quickly determined translations to be Taxicab isometries. The group then returned to the Object level for isometries when they concluded that Taxicab isometries would be the same isometries as those that are Euclidean. This statement then had the students consider whether or not rotations are elements of the set of isometries in Taxicab geometry. While one student internalized that only certain specific rotations would be elements of the set of isometries in Taxicab geometry, the other students needed to use an example to help them to verify this claim. This result shows that the group needed the proposition to be proven at the Action level by a student who was at the Process level of rotations. After the students finished working with the rotation transformation, then they started considering the reflection transformation. When trying to determine whether or not reflections are isometries, the students displayed an Action level for the concept of reflections. Because the class period had ended, the students did not have the opportunity to continue their discussion on reflections.

From an analysis of what the students displayed, it appeared that group one students would start tasks by internalizing that specific triangles would be congruent or not congruent depending on geometry, but after testing, they considered that congruence would actually have to do with transformations rather than with what type of triangle they were using. This idea as well lead to testing it and later to generalizing transformations by considering that the set of Taxicab isometries could be a proper subset or possibly the complete set of Euclidean isometries. The students were in the process of reinforcing these considerations through the verification of specific isometries. It is unclear if the students would have had a solid definition of congruence with respect to isometries since they ran out of class time, but it appears that they were in midst of concluding that specific rotations, reflections, and translations would be Taxicab isometries.

Group Eight

After immediately completing Task 1a with a similar conclusion as that of group one, a member of Group Eight considered how orientation would affect distance in Taxicab geometry. This conclusion displays that the student had internalized that orientation was a factor in congruence and, therefore, the student had a Process level understanding of congruence; however, the student needed to use visual examples to help classmates in agreeing to this idea, showing that the rest of the group was at the Action level of congruence. This discussion then led the students to making examples of rotations by 90 degree increments, which, in turn, helped them internalize that specific rotations preserve Taxicab distance. In this example, the students went through the Action level of rotations to comprehend rotations at the Process level. Finally, this example led to their labeling specific rotations to be isometries.

Since the students had only been working examples with triangles, they believed that anything they classified as an isometry was only an isometry for triangles. Their conclusion to this demonstrated that the group was at an Action level of isometries since they were not able to internalize the effect of these transformations on objects other than triangles; however, a student raised the question if this conclusion was really only true for triangles. That question encouraged the students to extend their understanding of isometries beyond triangles and including arbitrary figures. From their examples using arbitrary figures, the students concluded that only specific transformations would be considered Taxicab isometries. To reassure themselves, the students decided to test transformations again on different figures displaying that they needed to work at an Action level to justify their reasoning, at least to themselves. After testing transformations, a student asked if they should test rotations by 30 degrees but the group members decided that that suggestion would not be Taxicab isometries but they did not say why. Therefore, it is unclear if the students had internalized that 90 degree increments of rotation would be the only rotations that would be isometries in Taxicab geometry or because they believed that 90 degrees were the only angles possible in Taxicab geometry. There is evidence to suggest that the group had a belief that objects could not be rotated by angles that were not 90 degrees when one student said, “So Taxicab only works with 90 degrees. We can’t rotate it by anything else?” followed by a confirmation from another group member.

Group 8 displayed that they were starting to internalize rotations as isometries, but to give themselves the confidence that this was a true hypothesis, the group rotated an object, which displayed an Action level of understanding with regard to rotations being isometries in Taxicab geometry.

Group Nine

Similar to the Groups One and Eight, Group Nine students had already declared that right triangles with congruent parts would satisfy the conditions of Task 1a, displaying a Process level of congruence. At this point, the students still had not considered orientation or how any other transformation would affect distance. They did proceed to measure their results to confirm their assumption. When the students continued onto Task 1b, they once again considered a specific triangle. This time, the students used an isosceles triangle (more specifically, what they considered to be an isosceles triangle in Euclidean geometry). They chose to use this visual representation knowing that in Euclidean geometry, two sides were congruent and a third may or may not be congruent to the other two, once again displaying that they had a Process level of congruence. After validating their decision to use isosceles triangles in their example, they decided to consider how orientation would affect distance.

Through their consideration, they showed a Process level of thinking, but they did not have much confidence in their assumption, so this lack of confidence led one of the students, to use Action level examples to convince the other students that orientation will affect distance in Taxicab geometry. This use of examples then caused the group to agree that orientation does affect distance and that they could use this knowledge to complete the second and third tasks. In the third task, the students concluded that it was possible to have a pair of triangles congruent in Taxicab geometry but not in Euclidean geometry so long as their sides were congruent even though their angles were obviously (by visual inspection) not congruent. This conclusion showed that the students had a Process level again for congruence, but congruence did not rely on isometries, but on the type of triangle they used.

When considering what transformations were isometries in Taxicab geometry, the students internalized their work in the previous three tasks to conclude that rotations are not isometries based on their knowledge that rotations change orientation and, therefore, must affect distance, thereby contradicting the belief that rotations are isometries. The group also noted that translations should be isometries because translations never affect the orientation of an object, so they will not affect distance either. This display of reasoning shows that the students were at a Process level for both translations and rotations. Similar to what group eight did, the students of this group Nine also decided to test reflections to determine if they were isometries or not through using examples, thereby displaying that the students were at an Action level for reflections.

When determining how something would be defined as congruent in Taxicab geometry, the students first indicated the desired answer, saying that for two objects to be congruent in Taxicab geometry, there must be a series of Taxicab isometries that lead from one congruent object to the other congruent object. This answer was then revoked since the students noticed that they had did not considered the importance of angles in the third task and therefore, they falsely concluded that isometries have nothing to do with congruence.

It was evident that many of the group's conclusions came from their experiences working through the series of tasks. If they arrived at a conclusion in Task 1 but the conclusions they reached in Tasks 2 or 3 did not agree with what they found in Task 1, then the group believed that what they found in Tasks 2 or 3 to be wrong. They did not seem to consider that they could amend their original deductions. This thought process displays that the students internalize from what they have done previously and forget to experiment in the Action level to improve their schemas for their concepts. In other words, the students are reluctant to challenge their original findings.

Conclusion

Overall, the groups displayed that they had a tendency to start Task 1 by considering types of triangle pairs rather than the considering what would affect distance. The second task caused the students either to consider triangles once again or to consider orientation. Nonetheless, they still did consider how orientation affects distance before arriving at the third task. Throughout the completion of these tasks, at least one of the students in each group had an internalized idea of how orientation would affect the distance of objects in Taxicab geometry. Those that had this understanding gave an example to group members in order to convince them of the relationship between orientation and distance. It was this back and forth between one student internalizing an idea whether it be about how orientation affects distance or what an isometry is, and that same student trying to convince group members through examples and visual aids that led to the reinforced of these ideas on the groups' overall understanding of these concepts.

From the analyses conducted, the researcher observed that group nine arrived at a definition for congruence with respect to isometries. They determined that two objects were congruent in Taxicab geometry if there exists a series of Taxicab isometries that will superpose one congruent object on to its congruent object. Students of Group Nine later revoked this definition of congruence because their work on the third task contradicted their work in earlier tasks. This revocation demonstrates that Group Nine consider their conclusion to the third task to have more credibility than their original definition of congruence. This conclusion could be due to the fact that the students dedicated more time to the first task and, therefore, felt more comfortable with this answer as opposed to their initial definition of congruence with respect to isometries. This result displays the Action level of congruence.

It can be seen with all three groups, the students internalized a concept that they were studying, whether it be isometry or congruence. They started the tasks with a Process level of their congruence concept that became unpacked and restructured when Taxicab geometry was introduced. The constant change in levels between Action, Process, and Object helps the groups in changing how they view each concept. Group nine displayed that while at first the members had the expected view of what congruence in Taxicab geometry is, their lack of time to bring this concept to an Action level ultimately kept the group from being convinced of the correct view of the congruence concept.

Discussion

A limitation for this project was the use of archived data. The tasks assigned to the groups were designed for a different research project. If we seek to answer how student learning of congruence with respect to isometries follows APOS theory, the tasks need to be designed with that trajectory in mind. This researcher is currently collecting data under such a study design. Finally, most groups did not complete all assigned tasks in this assignment, so generalizing from the three cases presented is challenging.

If future research were to replicate this work, collection of data could be handled outside of a classroom where there were no time constraints to complete the tasks, thereby collecting the conclusions of all the students as opposed to only those students who completed the assignment. Additionally, if the tasks were based on the primary investigator's expected learning trajectory for the congruence definition, the students might arrive at the desired definition for congruence.

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