Demystifying Special Relativity; the K-Factor

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ABSTRACT

This paper targets upper-level high school students and incoming college freshmen who have been less exposed to Special Relativity (SR). The goal is to spark interest and eliminate any feelings of intimidation one might have about a topic brought forth by world genius Albert Einstein. For this purpose, we will introduce some ideas revolving around SR. Additionally, by deriving the relationship between the k-factor and relative velocity, we hope that students come to an appreciation for the impact of basic mathematical skills and the way these can be applied to quite complex models. Advanced readers can directly jump ahead to the section discussing the k-factor.

keywords : relative velocity, world lines, space-time

1 Introduction

1.1 A Morsel of History

Throughout time, scientists have searched for 'just the right' model comprised of laws, assumptions and theories that could explain our physical world. One example of a leading physical law is: All matter in the universe is subject to the same forces. It is important to note, this principle is agreeable to scientists, not because it is empirically true, but because this law makes for a good model of nature. Consequently, this law unifies areas like mechanics, electricity and optics traditionally taught separate from each other. Sir Isaac Newton (1643-1727) contributed to this principle with his 'three laws of motion.' They were so mathematically elegant, that the Newtonian model dominated the scientific mind for nearly two centuries. Yet, like all models, Newtonian physics had its limitations; a number of phenomena, such as a those related to light, electricity and magnetism could not be explained. For example, Descartes (1596–1650) first proposed the existence of an ether (field) to explain the interactions between magnets and iron nails. He suggested that all matter resided within this ether, and that magnetic waves explained these interactions. Newton himself added to this proposition suggesting the existence an ether containing gravitational force. The ether concept further grew to include propagating light waves. All these suggestions of waves in an ether seemed natural, and working out their behavior mathematically was relatively simple. However; thus far, there was no experimental evidence that confirmed the waves behaved as expected. It was not until the Michelson-Morley experiment of 1887 which failed to confirm the existence of such an ether, that forced dumbfounded physicists and mathematicians to reconsider the model they were using ([4], p.9). Enter Albert Einstein, who solved the Michelson-Morley puzzle with a paper entitled "The electrodynamics of moving bodies" (see[3]). What sprouted out of this paper his first theory of relativity. At the time, his explanation seemed to contradict Newton ([4], section 1.6). Today, Einstein's Special Relativity is a theory that supersedes Newton's three laws of motion, including objects travelling at/close to the speed of light. D'Inverno warns us not to call scientific models like the Newtonian incorrect. Rather, he urges us to say that Newton's model is a good model within its "range of validity" ([2], p.16).

1.2 Postulates of SR

What makes special relativity 'special'? In SR, we only consider situations where observers are moving at a constant speed relative to each other and do not feel any inertial forces, such as centrifugal force. Moreover, gravitation is not taken into account. SR allows any object moving in this way to be its own **inertial observer**. Additionally, all inertial observers are equipped with their very own reference frame, A.K.A. inertial frame (IF), and a clock (see Fig. 1). Einstein chose to found SR on two physical postulates:

Postulate I: All inertial reference frames are equivalent.
Suppose an astronaut and a table were floating in a dark void. The only light present is the light emanating from the two bodies so that the astronaut can see himself and the table but nothing else. The astronaut observes the table approaching him. But, what can be set about his own motion? Without a floor, or fixed absolute frame of reference, who's to say the astronaut is stationary? This thought experiment unravels many more questions. Is the table stationary and the astronaut moving closer? Are they both

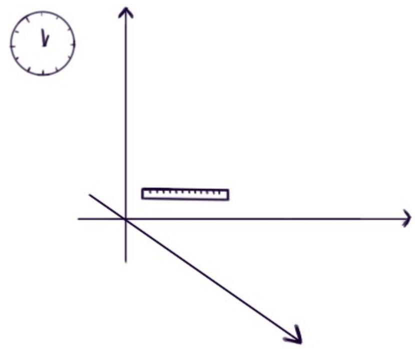

Figure 1. The IF is comprised of three orthogonal axes of equal unit lengths, a graduated ruler and a clock.

only moving closer to each other? Or are they moving closer together as they are moving in space? Postulate I asserts that all these viewpoints are equally valid, provided the astronaut and the table approach each other with constant velocity.
Postulate II: The speed of light is constant and its velocity is invariant.
Postulate II has two implications. Imagine two inertial observers A and B. Consider this scenario from A's reference frame. A is stationary and B is moving past A at some constant speed. B lets off a laser beam in the direction he is heading. Firstly, Postulate II implies, light (in a vacuum) travels a constant speed. (we denote a relative constant c = 1). Secondly, the speed of light is c in both A and B's IF.

2 The Notion of Time in Special Relativity

> At their core, Einstein's relativity theories (both the special theory of 1905 and the general theory of 1915) are the modern physical theories of space and time, which have replaced Newton's concepts of absolute space and absolute time by Spacetime ([4], p.3).

Instead of going in to what absolute space and time are and how they compare relativistic space-time, we are going to simply throw away any perception of time and space and focus on altogether new way to like at time using world lines. Just know that, as humans, we don't live in space time, our understand of space and time come from the natural consequence of existing as slow moving observers (who are non-inertial). the reader advised to expend his energy grasping this idea, but if if he were to insist he fair better to know that understand space-time is an imaginative pursuit.

2.1 A Brief Explanation of World Lines

World lines can help us understand relativistic time. For simplicity, we assume that only one spatial direction is relevant (say the x-axis). A world line is the trajectory of an object A, drawn in a two-dimensional coordinate system where the horizontal axis indicates space (x) and the vertical axis indicates time (t). (t,x) are measured with respect to an arbitrarily chosen observer. Below are some examples of space time diagrams as shown from A's perspective. Note that the faster an object moves, the more tilted its world-line becomes. We shall assume that nothing moves faster than the speed of light, so the angle between a world-line and the time axis is always less than 45 degrees.

In Fig. 2, an observer A is at a point in space as shown on the x-axis. In the space time diagram, a stationary observer A looks like vertical line. In the interest of brevity and active learning, three illustrations are provided below to guide the understanding of space-time scenarios. Figures 3, 4, 5 are depictions of two inertial observers A and B moving through space in A's perspective of time.

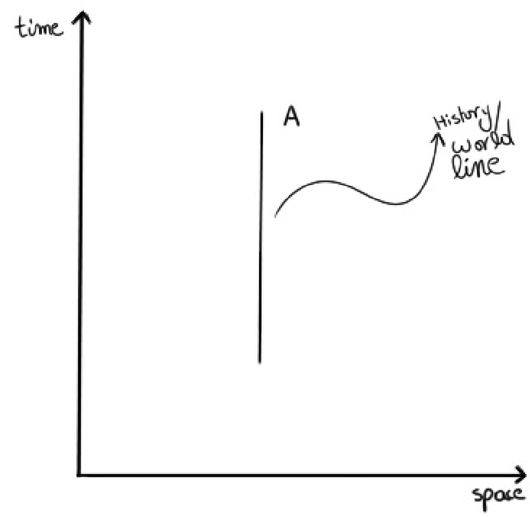

Figure 2. Space-Time Diagram showing A's trajectory in time

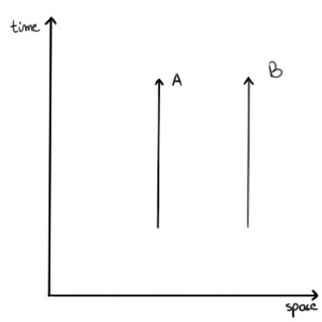

Figure 3. Space-time diagram showing inertial observers A and B stationary, or moving at the same velocity relative to each other.

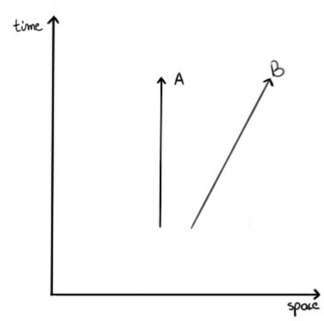

Figure 4. Space-time diagram shown form A's reference frame. B is moving away from A, moving to the right of A.

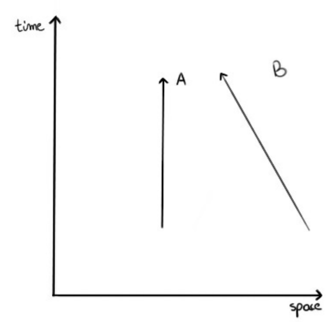

Figure 5. Space-time diagram shown form A's reference frame. B is moving closer to A from the right.

In the next sections, we will see how world-lines will be elementary in finding the K-factor, and deriving its relationship with relative velocity.

2.2 K-factor and Time Dilation

We will look at a problem adopted from Herman Bondi's "Special Relativity and Common Sense"Bondi: p.78 to bring about the idea of the K-factor. Consider a scenario such as that in Figure 6. According to A's IF, B is moving away from A to the right. Suppose A emits a light beam to B every 6 minutes. For a suitable velocity of B, B intercepts the beams every 9 minutes. Geometrically speaking, the reader might agree with the supposition that the 6 and 9 min intervals are repeated without further questioning. For the skeptical reader, we would like to explain this further. Since light travels at constant speed, (and because A and B are inertial observers) the first beam, beam 1 A will take some time to reach B. Since the speed of light is of course faster than B, a finite period of time passes before beam 1 reaches B. Meanwhile, A is timing 6 min before A emits a second beam, beam 2. If we look at the history so far, and since we know $Speed_{beam2} = Speed_{beam1} = Speed_{light} = c = 1$, then the lines of propagation of beam 1 and beam 2 are parallel (always 45° from the horizontal axis).

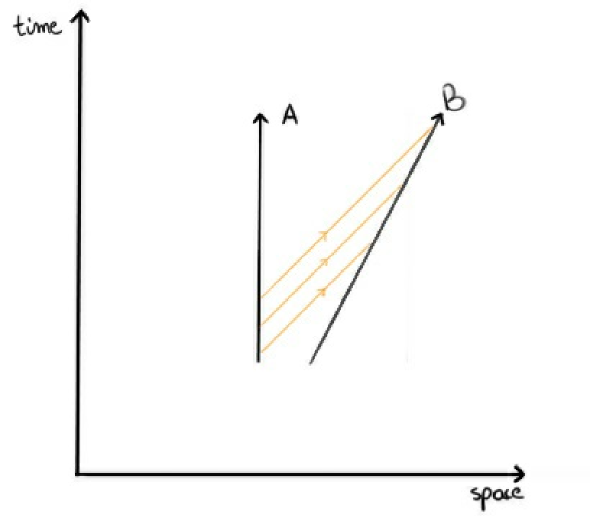

Figure 6: Space-time diagram showing interval of emission from A (I_e) and an interval of reception to B (I_r)

Imagine B moving farther away from A, thus, after 6 minutes, B is farther and farther away from A, while A remains in the exact same point in space. Since beam 2 has the same speed c, and B is farther away now, it follows that beam 2 reaches B after a longer time. We leave the discussion of the opposite scenario, where B is approaching A, to the reader. It is left to the reader to think up the converse of such a scenario by comparing the interval of transmission with interval of reception of B that is approaching A with respect to A. The Figures below are designed to guide the reader into building an intuitive sense of the size of intervals I_e and I_2 in relation to B's relative velocity.

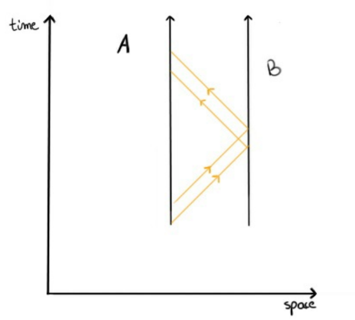

Figure 7. Shows relative velocity zero. Here the K-factor is 1.

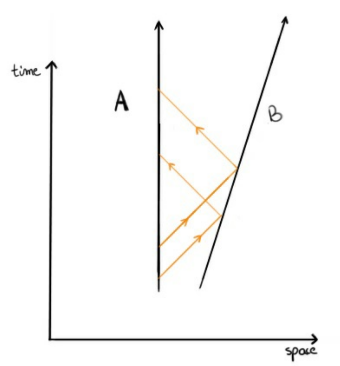

Figure 8. Shows positive velocity of B relative to A (distance between the worlds is growing over time). Here, the K-factor is < 1.

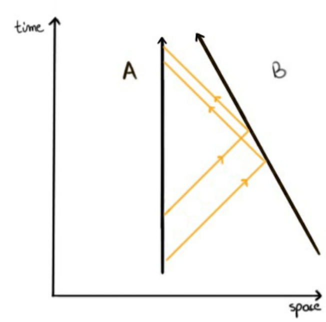

Figure 9. Shows negative velocity of B relative to A (approaching A, direction is negative and the distance between the world-lines is shrinking). Here the K-factor is > 1.

According to Hermann Bondi[1] (p.88), all the effects of Special Relativity can be conveniently derived from the concept of the K-factor. To introduce the concept, imagine that both A and B are carrying clocks. Whenever we are in the situation that A sends out two successive flashes of light such that the time $t > 0$ is the interval on A's clock, B can measure the time interval between the reception of the first flash and the second flash on B's clock. We define K to be the ration between the two time intervals

$$K = \frac{I_r}{I_e} = \frac{\text{Time elapsed for B between reception of the two beams}}{\text{Time elapsed for A between the emission of the beams}}$$

The following exercises are to help the reader to infer that, looking on from A's perspective, the K-factor relates the time interval (measured with A's clock) of two signals departing at A and the time interval of their arrival at B (measured with B's clock). First assume that the two observers fly away from each other and that their relative velocity leads to a k-factor of $k = \frac{3}{2}$.

(a) If A sends beams at 6 min intervals according to A's clock, what is interval of reception on B according to B's clock?

$$\frac{3}{2} \times 6 = 9$$

B will receive the signals 9 min apart.

(b) If A sends laser beams at 12 min intervals according to A's clock, what is the interval of reception on B according to B's clock?

$$\frac{3}{2} \times 12 = 18$$

B will intercept A beams every 18 min.

Next assume that the two observers are approaching each other with the same velocity magnitude as before. It turns out that in this case, the k-factor is the inverse of the one in (a) and (b), namely $k = \frac{2}{3}$.

(c) If B sends flashes to A in 18 min intervals (according to A's clock), what is the interval of reception at A in A's time?

$$\frac{2}{3} \times 18 = 12$$

A will receive the the flashes in 12 min intervals on A's clock.

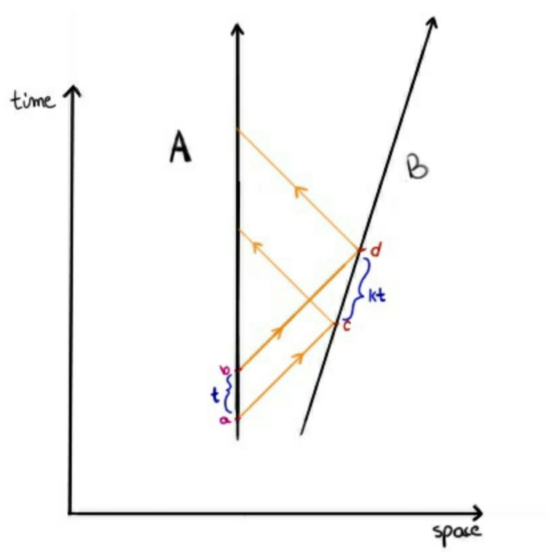

Figure 10. Space-time diagram showing $I_r = Kt$ at B depending on $I_e = t$ in A. We say this is $K_A = \frac{I_r}{I_r}$ is the k-Factor of A.

We postulate that the factor of proportionality K between the two time intervals depends only on the relative velocity of A and B. In other words, if A sends two beams to B with a lapse of t between them, then B will measure the time interval Kt between the reception of the first beam and the reception of the second beam.

3 Relationship between K-value and Velocity

The key point of our discussion is to show that the precise form of the K-factor in A's IF, K_A, can be determined:

$$K_A = \sqrt{\frac{1+v_{AB}}{1-v_{AB}}},$$

where v_{AB} is the relative velocity of B with respect to A. We will take $v_{AB} > 0$ to mean that B is receding from A, i.e. the distance between A and B is growing. Analogously, $v_{AB} < 0$ means that B is approaching A, i.e. the distance between A and B is shrinking over time. The question: '... over who's time?' might arise. The natural answer is the time read from A's clock.
Referring to Figure 11 below, consider the case when Brandon (B) and Adrian (A) are two inertial observers that started from the same point O in space (neglect gravitational forces). Over time, B moves away from away A at a constant rate. Where the world-lines of A and B met, the two observers synchronized their clocks. That is, they reset their clocks to show time equal to 0. Adrian and Brandon also agreed to perform an experiment whereby A waits an agreed-upon amount of time t before he emits a light beam at the point Q to B. B receives the beam at point P, holds up a mirror and reflects it back to A, who receives it at point R. Pay attention to the fact that we are working from Adrian's frame of reference (A's perspective).
In this case, t is both a moment in time (emission of the signal from Adrian) and a time interval (between the instant A and B pass each other and the emission of the signal: $|0-t| = t$). Adrian now deduces, using the definition of the K-factor, that Brandon should intercept the beam at time $K_A t$.

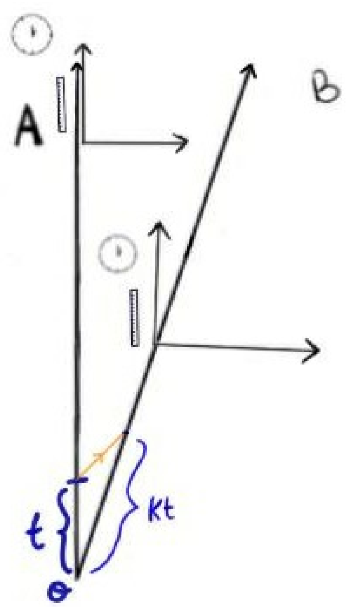

Figure 11. World lines showing inertial observers A and B meeting at point O, synchronizing their time. At O, both Adrian's (A's) and Brandon's (B's) clock show the time zero.

As agreed for the experiment, at time $K_A t$ on Brandon's watch, he anticipates the light beam by holding the mirror. At the moment it reaches Brandon, and is immediately reflected back to Adrian. For simplicity, assume there is no time lapse between interception and reflection of the light.
Applying K_A-factor again, the reflected light will reach Adrian again at $K_A(K_A t) = K^2 t$. (To help with our calculations, events such as the point of reflection are given letter names for clarity purposes.) An illustration of this experiment is found on Fig. 12.

Coincidentally, a triangle QPR is formed by the events of emission by A, reflection from B and reception of the signal by A. Because of some assumptions we have made in this paper about SR, together with basic euclidean geometry, we discover that

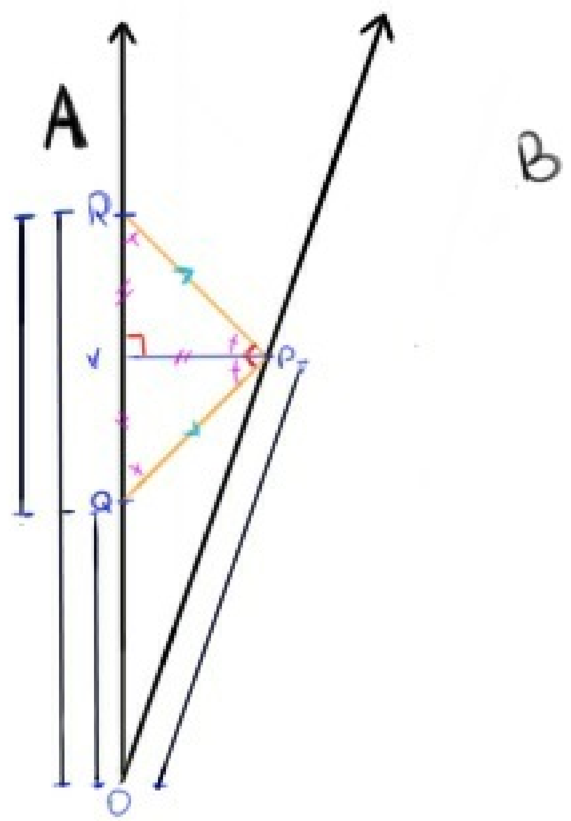

Figure 12. World lines showing A and B from A's perspective conducting a light experiment. Q is the event of emission, OQ is the interval of emission equal to t. P is the event and instance of reception and reflection, OP is the interval of this event and is equal to Kt. R is the event of reception from B to A (still in A's perspective, A.K.A. on A's clock), hence: QR= K^2t. V is the point in time on A's clock) where the light emitted from A reached B.

QPR is an isosceles triangle. Readers are encouraged to see this short and sweet proof found in Section 4. The interval QR, is the time (on A's clock) between emitting the light and receiving it back from B. QR can be written in terms of t as follows: QR= $K_A^2 t - t$. Let the time at which B reflects the incoming signal be denoted by V, as seen by Adrian. Recall properties of isosceles triangles. Since (VP) $\perp$ (QR), V is the midpoint of segment [QR]. Then the interval between synchronizing and reflection is

$$OV = \frac{1}{2}QR + OQ = \frac{1}{2}(K_A^2 - 1)t + t = \frac{1}{2}(K_A^2 + 1)t.$$

So, the time elapsed (on A's clock) for the light to reflect by B is $\frac{1}{2}(K_A^2 + 1)t$.
Next, let's find the distance between A and B at the time of reflection. This distance would be equivalent to the length of the segment [VP]. Once again, geometric properties get the spotlight. We have VP = VQ = VR (See Section 4).

So, VP = VQ = $\frac{1}{2}(K_A^2 - 1)t$.

We use the basic definition of velocity and adapting it to fit our idea of relative velocity of B to A

$$V_{AB} = \frac{\text{distance travelled from A}}{\text{time on A}}.$$

We have therefore

$$v_{AB} = \frac{\text{Distance}}{\text{Time}} = \frac{VP}{OV} = \frac{\frac{1}{2}(K_A^2 - 1)t}{\frac{1}{2}(K_A^2 + 1)t}.$$

We can simplify this expression further

$$v_{AB} = \frac{\frac{1}{2}(K_A^2 - 1)T}{\frac{1}{2}(K_A^2 + 1)T} = \frac{(K_A^2 - 1)}{(K_A^2 + 1)},$$

and then solve for K_A:

$$\begin{aligned}
& v_{AB}(K_A^2 + 1) = (K_A^2 - 1) \\
\iff\quad & v_{AB}K_A^2 + v_{AB} = K_A^2 - 1 \\
\iff\quad & -K_A^2 + v_{AB}K_A^2 = -1 - v_{AB} \\
\iff\quad & -K_A^2(1 - v_{AB}) = -(1 + v_{AB}) \\
\iff\quad & K_A^2 = \frac{(1 + v_{AB})}{(1 - v_{AB})} \\
\iff\quad & K_A = \sqrt{\frac{(1 + v_{AB})}{(1 - v_{AB})}}
\end{aligned}$$

which is our desired expression for the k-factor.

4 Geometric Proof of the Triangle QRP being isosceles

Given:
1)Postulate II, then $\measuredangle VQP = \measuredangle VPR$
Additionally,
2) $\measuredangle VQP = \measuredangle VPR = 45^\circ$ (Assumption under SR) Since
3) A inertial observer, then the history is straight line, and
4) We assumed to be in A's perspective, then the world line is vertical,
Then we have that (VP) $\perp$ (QR), so $\measuredangle VPQ = 45^\circ$ (sum of angles in the triangle QPV must equate 180°)
Now,

$$\measuredangle QPR = QPV + VPR = 90^\circ$$

So, in triangle QPR, $\measuredangle QPR = 45^\circ$ (sum of angles in a triangle)
Therefore, QPR is an isosceles triangle.

5 Conclusion

Newton's Three Laws of motion were expanded to include Einstein's two postulates of SR. This resulted in important thought experiments, one of which we covered in this paper (relationship of K-factor to relative velocity). Our aim was to encourage students to read more about SR and create their own thought experiments about questions that inspire them. Many of the mathematicians we look up to were also philosophers. It is remarkable to see the great thinkers of history using mathematical models to think about their theories. What might be most important to take away from this manuscript is that mathematics, even in its simplest tools, contains power far beyond one could expect.
After looking into some of the mathematics in this paper, we can now appreciate a relativistic view of the universe and ask further questions such as, mathematically, how do we justify rejecting the negative result of taking the square root in the final

stages of deriving v_{AB} from K_A in section 3. The reader can start thinking about what it means to have the negative result rather than the positive - or if the two could somehow be equivalent.
It may serve well to mention, that there is plenty more to be said about SR and relativity. Since this paper had to be condensed, many important concepts fell through in our light exploration of the subject. For this reason, readers are encouraged to view the reference section for good reading material. Ultimately, we hope this paper ignited your curiosity to go further.

Acknowledgements

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Additional Information

Source of Figures. All figures were replicated from D'Iverno but produced originally for this paper using the software Autodesk Sketchbook version 1.8.2 created by Autodesk Inc, copyright 2018. **Conflict of interests**. Conflict of interest statement (or statement of competing interests) can be found as a sample policy from I.R.S. website Conflict of interests policy

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